Solutions Exam Analysis November 3, 2014

1. (a) Since A is bounded there exists M' such that $|a| \leq M'$ for all $a \in A$. (**1 pt.**).
   Let $l \in L$. Take $\epsilon = 1$ then there exists $a \in A$ such that $|l - a| < 1$. (**2 pt.**)
   Hence $|l| < |a| + 1 \leq M' + 1 =: M$, and thus L is bounded (**2 pt.**)

   (b) Take $\epsilon > 0$. Since $c = \sup L$ there exists $l \in L$ such that $c - \frac{1}{2}\epsilon < l$. (**2 pt.**)
   Since $l \in L$ there exists $a \in A$ such that $|l - a| < \frac{1}{2}\epsilon$. (**1 pt.**)
   Hence

   $$|c - a| \leq |c - l| + |l - a| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

   Hence c is a limit point of A. (**2 pt.**)

   (c) Let $\epsilon > 0$. Suppose there were infinitely many elements $x \in A$ with $x > c + \epsilon$. Then there would exist a limit point l of A (and thus $l \in L$) with $l \geq c + \epsilon$, which contradicts $c = \sup L$. (**3 pt.**)
   On the other hand, since $c = \sup L$ there exists an $l \in L$ such that $c - \frac{1}{2}\epsilon < l$. Furthermore, since $l \in L$ there exist infinitely many $x \in A$ such that $|l - x| < \frac{1}{2}\epsilon$. It follows that there exist infinitely many $x \in A$ with $c < l + \frac{1}{2}\epsilon < x + \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$, showing that there are infinitely many $x \in A$ with $x > c - \epsilon$. (**2 pt.**).

2. (a) Consider $x < y \in \mathbb{R}$ and let $x \neq 0$. Then there exists $t \in \mathbb{R}$ such that $y = tx$. Hence for $x \neq 0$

   $$f(y) - f(x) = f(tx) - f(x) = (t - 1)f(x) = (y - x)\frac{f(x)}{x}$$

   This implies that f is continuous at every $x \neq 0$. (**4 pt.**)
   Finally, for $x = 0$ it follows that $f(0) = f(0 \cdot 0) = 0 \cdot f(0) = 0$, and hence $f(y) - f(0) = f(y) = yf(1)$. (**2 pt.**)
   Thus for any ϵ we can take $\delta = \frac{\epsilon}{|f(1)|}$, and $|y| < \delta$ will imply $|f(y) - f(0)| < \epsilon$, proving continuity at $x = 0$. (**2 pt.**)

   (b) Take $x \neq 0$. Then

   $$\frac{f(tx) - f(x)}{tx - x} = (t - 1)\frac{f(x)}{x}$$

   and it follows that $\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \frac{f(x)}{x}$, proving differentiability at any $x \neq 0$ and $f'(x) = \frac{f(x)}{x}$. (**5 pt.**)
   For $x = 0$ we obtain

   $$\frac{f(y) - f(0)}{y - 0} = \frac{f(y)}{y} = f(1)$$

   for all $y \neq 0$, and thus $f'(0) = f(1)$. (**2 pt.**)

3. Consider $h(x) := g(x) - f(x)$. Then $h(a) \geq 0$ and $h'(x) > 0$ for $x \in (a, b)$. Let $y \in (a, b]$. Then by the Mean Value theorem on the interval $[a, y]$ there exists $x \in (a, y)$ such that

   $$h(y) - h(a) = h'(x)(y - a)$$

   Hence, $h(y) = h(a) + h'(x)(y - a) > 0$.

4. Write

$$|g_n(f_n(x)) - g(f(x))| \leq |g_n(f_n(x)) - g(f_n(x))| + |g(f_n(x) - g(f(x))|$$

(2 pt.)
Let $\epsilon > 0$. Since $g_n \to g$ uniformly there exists N_1 such that for all $n \geq N_1$

$$|g_n(y) - g(y)| < \frac{1}{2}\epsilon,$$

for all y, and thus also for all $y = f_n(x)$. **(4 pt.)**
By uniform convergence of g_n it follows that g is continuous, and thus uniformly continuous on $[c, d]$. Hence, there exists δ such that $|g(z) - g(y)| < \frac{1}{2}\epsilon$ for all y, z with $|z - y| < \delta$. **(4 pt.)**
Since $f_n \to f$ uniformly it follows that there exists N_2 such that for $n \geq N_2$ we have $|f_n(x) - f(x)| < \delta$ for all x, and hence

$$|g(f_n(x) - g(f(x))| < \frac{1}{2}\epsilon$$

for all x. Now take $N := \max\{N_1, N_2\}$, and use the first inequality. **(5 pt.)**

5. (a) Since $|1 + nx^2| \geq an^2$, by the Weierstrass Test the series converges uniformly on each interval $[a, \infty)$.

   (b)

   $$f_n'(x) = \frac{n^2}{(1 + n^2x)^2}$$

   Since $(1+n^2x)^2 \geq a^2n^4$ it follows that $\frac{n^2}{(1+n^2x)^2} \leq \frac{1}{a^2n^2}$, and thus by the Weierstrass test the series $\sum f_n'(x)$ converges uniformly on any interval $[a, \infty)$, and hence the series $\sum f_n(x)$ is differentiable.

6. (a) The function $F(x) = \int_0^x f$ is continuous. **(3 pt.)**
   Hence by the Intermediate Value theorem on the interval $(0, 1)$, and the fact that $F(0) = 0$, $F(1) = 2$, and $0 < 1 < 2$, there exists $x \in (0, 1)$ such that $F(x) = 1$. **(6 pt.)**

   (b) Every subinterval of every partition P always contains an element of $\mathbb{Q}$ and an element not in $\mathbb{Q}$. Hence for every partition P we have $U(f, P) = 1$ while $L(f, P) = -1$. Hence f is not Riemann integrable. **(6 pt.)**